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On Decay of Correlations for Unbounded Spin Systems with Arbitrary Boundary Conditions

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We propose a method based on cluster expansion to study the truncated correlations of unbounded spin systems uniformly in the boundary condition and in a possible external field. By this method we study the spin–spin truncated correlations of various systems, including the case of infinite range simply integrable interactions, and we show how suitable boundary conditions and/or external fields may improve the decay of the correlations.

KEY WORDS: Unbounded spin systems; decay of correlations; cluster expansion.

INTRODUCTION

In recent times a considerable effort has been spent to generalize the classical framework of the complete analiticity for bounded spin systems to the unbounded case. This effort is motivated by the fact that, both in the bounded and in unbounded case, it is in general difficult to prove directly the log-Sobolev inequality, which ensures the complete analyticity, or the existence of a spectral gap for the spin systems, while it is possible to prove the equivalence of the existence of the spectral gap with some other property of the systems easier to check.

The complete scenario of the bounded case, see refs. 1 and 2, has been almost completely recovered in the unbounded case in ref. 3, which proved that the log-Sobolev inequality and the existence of a spectral gap for models of interacting unbounded spins equipped by a local potential satisfying reasonable conditions and finite range interactions is equivalent

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to the exponential decay of the spin–spin truncated correlation uniformly in the boundary conditions.

In many other recent papers, see, e.g., refs. 4–8, the uniform exponential decay of the correlations has been proved with various techniques for the same range of models.

The decay of correlations for unbounded spin systems with empty boundary conditions is an argument studied since a long time, see, e.g., refs. 9–11.

A related topic is the study of the unicity of the Gibbs measure. Results in this sense have been obtained so far using suitable generalizations of the Dobrushin theory, see refs. 12 and 13 and, more recently, ref. 14, where the decay of the correlations is also treated. In refs. 13 and 14, however, only the finite range case is studied. Note, moreover, that in refs. 4–8 the problem is to find the behaviour of the truncated correlation independently on the boundary conditions, i.e., also when the boundary does not satisfies the conditions needed to prove unicity.

In this paper we propose a method to study the decay of the spin–spin correlations uniformly in the boundary conditions based on cluster expansion techniques. Such method is inspired by ref. 9.

With our technique we are able to prove all the known results for the finite range case, including the equivalence of the exponential decay of the spin–spin truncated correlation and the analogous decay of the truncated correlations between measurable functions. Moreover we can prove some additional generalizations.

In particular we prove the exponential decay of the correlations uniformly in the boundary conditions when the interaction in the system is infinite range with exponential decay. Then, for infinite range interactions with summable power law, we prove that the spin–spin correlations have the same decay as the interaction. This result was not known in literature even for empty boundary conditions (see reference above). In this case one expect from a dynamic point of view a slow convergence to the equilibrium. Very recently an analogous result has been proved with different techniques in ref. 15 for bounded spin models with ferromagnetic interactions. Note that with our technique we do not need to impose any constraint on the sign of the interaction, and hence we are able to prove the decay of the correlation even, e.g., for disordered systems with summable interaction.

Finally we show how large boundary conditions and/or the presence of an external field may improve the decay of the correlations. As a byproduct of this we are able to control the decay of the correlations and the convergence of the free energy at low temperature for systems with boundary conditions and constant external field suitably chosen.

The main idea behind our results is the following. Let us call $\mu(\phi_x)$ the product measure of the system and J the strenght of the interaction measured in the L_1 norm

$$\sup_{x \in \mathbf{Z}^d} \sum_{y \neq x} |J_{xy}| = J < +\infty \tag{0.1}$$

The quantites that one has to control in order to have convergence of the cluster expansion are the following moments of the modified product measure

$$C_\alpha(J) = \int |\phi_x|^\alpha \, e^{J\phi_x^2} \, d\mu_x(\phi_x) \tag{0.2}$$

where $\alpha \in \mathbf{N}$. In order to have convergent expansions one has to prove that

$$C_\alpha(J) \leqslant \alpha! \, A^\alpha C(J) \tag{0.3}$$

where the constants A and $C(J)$ are independent on the boundary conditions, and to impose that the quantity $JC(J)$ is small.

It is clear that when a dependence on the boundary conditions is included in the product measure μ_x, there is no hope to find a uniform bound of the form (0.3). Nevertheless one can introduce, simply shifting the fields, a different model in which the field is substituted by its deviation with respect to some configuration which minimizes the Hamiltonian. In this way the free energy is not uniformly bounded in the boundary conditions, but this divergence appears only in an overall constant factor, and will not affect the truncated correlations.

On the other side the new product measure that one obtains for the shifted fields is now under control in the sense of (0.3) uniformly in the boundary conditions.

Similar ideas have been used before in different context (see, e.g., refs. 16 and 17). Since the properties of the resulting shifted measure are quite crucial in our work, we describe it in some detail in Section 2 and we study its properties in Section 3.

The rest of the paper is devoted to the computation via cluster expansion of the spin–spin correlations exploiting (0.3).

This is in some sense quite standard (see the references above), however our careful treatment of the convergence of the series, using a representation of the spin spin correlation borrowed from Simon[18] and the Battle–Brydges–Federbush trees technique, allows us to obtain the above mentioned decay of the correlations with the same power as the interaction. The representation of the spin–spin correlation is presented in Section 4.

In Section 5 we state our main general result and we add to it some useful remarks and exemples. The proof of our theorem is in Section 6.

1. THE ORIGINAL MODEL

Let us denote with $\mathbf{Z}^d$ the simple cubic unit lattice in d dimensions equipped with the usual Euclidean distance.

Suppose that in each site $x \in \mathbf{Z}^d$ is defined a variable ϕ_x, called *spin* or *field*, which takes values in $\mathbf{R}$. A configuration $\bar{\phi}$ is a function $\bar{\phi}$: $\mathbf{Z}^d \to \mathbf{R}$. Let $\Lambda \subset \mathbf{Z}^d$. We call $\bar{\phi}_\Lambda$ the restriction of $\bar{\phi}$ to Λ. We also denote $\Lambda^c \equiv \mathbf{Z}^d \backslash \Lambda$.

We consider the lattice model described by the Gibbs measure

$$\mu_\Lambda^\omega(\cdot) = \frac{1}{Z(\Lambda, \omega)} \int \prod_{x \in \Lambda} d\phi_x \, e^{-H(\bar{\phi}_\Lambda, \omega)}(\cdot) \tag{1.1}$$

where $d\phi_x$ is the Lebesgue measure in $\mathbf{R}$ and the partition function $Z(\Lambda, \omega)$ is defined by

$$Z(\Lambda, \omega) = \int \prod_{x \in \Lambda} d\phi_x \, e^{-H(\bar{\phi}_\Lambda, \omega)} \tag{1.2}$$

The Hamiltonian of the system is

$$\begin{aligned} H(\bar{\phi}_\Lambda, \omega) &= \sum_{x \in \Lambda} U(\phi_x) - \sum_{\{x, y\} \cap \Lambda \neq \varnothing} J_{xy} \phi_x \phi_y + \sum_{x \in \Lambda} h_x \phi_x \\ &= \sum_{x \in \Lambda} [U(\phi_x) - \phi_x \omega_x] - \sum_{\{x, y\} \subset \Lambda} J_{xy} \phi_x \phi_y \end{aligned} \tag{1.3}$$

$U(x)$ is an even polynomial of degree $2k$, $k > 1$, of the form

$$U(x) = x^{2k} + \sum_{i=0}^{k-1} u_{2i} x^{2i} \tag{1.4}$$

with $u_{2i} \in \mathbf{R}$; as stated in the introduction, the pair potential J_{xy} is such that

$$\sup_{x \in \mathbf{Z}^d} \sum_{y \neq x} |J_{xy}| = J < +\infty \tag{1.5}$$

$h_x \in \mathbf{R}$ represents the external field, and in the last line of (1.3) we defined

$$\omega_x = -h_x + \sum_{y \in \Lambda^c} J_{xy} \phi_y \tag{1.6}$$

The boundary fields ϕ_y with $y \in \Lambda^c$ must be chosen in such way that $\sum_{y \in \Lambda^c} J_{xy}\phi_y$ is finite for all x.

In this paper we study the 2-points truncated correlation $\mu_\Lambda^\omega(\phi_x, \phi_y)$, defined by

$$\mu_\Lambda^\omega(\phi_x, \phi_y) = \mu_\Lambda^\omega(\phi_x \phi_y) - \mu_\Lambda^\omega(\phi_x)\, \mu_\Lambda^\omega(\phi_y) \tag{1.7}$$

As stated in the introduction, in the recent literature (see, e.g., refs. 3, 4, 6–8 for similar results), in the case of couplings J_{xy} small enough and finite range, the following bound has been proved

$$|\mu_\Lambda^\omega(\phi_x, \phi_y)| \leqslant C e^{-\gamma |x-y|} \tag{1.8}$$

with C and γ positive constants independent on Λ and ω. Moreover in ref. 3 it is proven the equivalence between (1.8), the log-Sobolev inequality and the existence of the spectral gap.

Here we propose an alternative technique to obtain the bound (1.8) which allows us to treat also the case of infinite range interactions, and, for suitable boundary conditions and/or external field the case of strong interactions.

2. THE SHIFTED MODEL

We write the Hamiltonian as in the last line of (1.3)

$$H = \sum_{x \in \Lambda} [U(\phi_x) - \phi_x \omega_x] - \sum_{\{x, y\} \subset \Lambda} J_{xy} \phi_x \phi_y \tag{2.1}$$

where ω_x is defined in (1.6).

Then we perform a change of variables defining new fields ψ_x which are simply a translation of fields ϕ_x, namely

$$\phi_x = \psi_x + \zeta_x \tag{2.2}$$

The translation vector $\bar{\zeta} = \bigcup_{x \in \Lambda} \zeta_x$ is chosen in such a way that it minimizes the Hamiltonian (2.1). Hence we define $\bar{\zeta}$ as a solution of the following set of equations

$$U'(\zeta_x) - \omega_x - \sum_{y \in \Lambda:\, y \neq x} J_{xy} \zeta_y = 0 \qquad \forall x \in \Lambda \tag{2.3}$$

Note that the system (2.3) always admits real solutions, since H is a polynomial of degree $2k$ in $|\Lambda|$-variables bounded below.

The Hamiltonian (2.1) may now be rewritten defining

$$q_x(\psi_x) = U(\psi_x + \zeta_x) - U(\zeta_x) - U'(\zeta_x)\,\psi_x \tag{2.4}$$

as

$$H = \sum_{x \in \Lambda} q_x(\psi_x) - \sum_{\{x, y\} \subset \Lambda} J_{xy}\psi_x\psi_y + C(\bar{\zeta}) \equiv \bar{H}(\bar{\psi}) + C(\bar{\zeta}) \tag{2.5}$$

where by (2.4) it is easy to see that $q_x(\psi_x)$ does not contain terms linear in the field for any x, and where

$$C(\bar{\zeta}) = \sum_{x \in \Lambda} (U(\zeta_x) - \zeta_x\omega_x) - \sum_{\{x, y\} \subset \Lambda} J_{xy}\zeta_x\zeta_y$$

can be bounded by a suitable constant of the form $|C(\bar{\zeta})| \leqslant |\Lambda|\, C$, where C depends in general from boundary conditions and the external magnetic field, and may diverge with them. The shift constants ζ_x in general depends on the boundary spin configurations and/or on the external magnetic field, and they can be arbitrarily large for any x (even inside the bulk) if the boundary fields and/or the external magnetic field are large enough. In general the choice of the configuration $\bar{\zeta}$ is not even unique. However the basic feature of the shifted Hamiltonian, i.e., the absence of linear terms in the field, (see next section for more details) is preserved for every choice of the local minimizer, and this is the essential feature to control uniformly the quantites of the form (0.2) in the sense of (0.3). An optimal choice of $\bar{\zeta}$ gives the optimal condition on the smallness of the interaction J_{xy} needed to have decay of correlations.

We define also the number

$$\zeta = \inf_{x \in \Lambda} |\zeta_x| \tag{2.6.}$$

and the constant ζ may be taken as a reasonable parameter to measure the influence of the boundary conditions and the external field on the system.

The partition function can be rewritten

$$Z(\Lambda, \omega) = e^{-C(\bar{\zeta})} \int \prod_{x \in \Lambda} d\psi_x\, e^{-\bar{H}(\bar{\psi}_\Lambda, \omega)} \tag{2.7}$$

Defining the local probability measure

$$dv_\omega(\psi_x) = \frac{e^{-q_x(\psi_x)}\, d\psi_x}{\int_{\mathbf{R}} e^{-q_x(\psi_x)}\, d\psi_x} \tag{2.8}$$

and defining also

$$\tilde{\mu}_{\Lambda}^{\omega}(\cdot) = \frac{\int \prod_{x \in \Lambda} d\nu_{\omega}(\psi_x)\, e^{\sum_{\{x,y\} \subset \Lambda} J_{xy} \psi_x \psi_y}(\cdot)}{\int \prod_{x \in \Lambda} d\nu_{\omega}(\psi_x)\, e^{\sum_{\{x,y\} \subset \Lambda} J_{xy} \psi_x \psi_y}} \tag{2.9}$$

it is easy to check that

Lemma 1.

$$\mu_{\Lambda}^{\omega}(\phi_x, \phi_y) = \tilde{\mu}_{\Lambda}^{\omega}(\psi_x, \psi_y) \tag{2.10}$$

Moreover the partition function (1.2) is written as

$$Z(\Lambda, \omega) = C_{\omega}(\Lambda)\, \tilde{Z}(\Lambda, \omega) \tag{2.11}$$

with

$$\tilde{Z}(\Lambda, \omega) = \int \prod_{x \in \Lambda} d\nu_{\omega}(\psi_x)\, e^{\sum_{\{x,y\} \subset \Lambda} J_{xy} \psi_x \psi_y} \tag{2.12}$$

$$C_{\omega}(\Lambda) = e^{-C(\bar{\zeta})} \prod_{x \in \Lambda} \int_{\mathbf{R}} e^{-q_x(\psi_x)}\, d\psi_x \tag{2.13}$$

3. PROPERTIES OF THE LOCAL MEASURE ν_ω

In order to bound by cluster expansion techniques the quantity $\log \tilde{Z}(\Lambda, \omega)$ and the truncated correlations we need to control uniformly in ζ and hence in ω the quantities

$$C_{\alpha}(J) = \int |\psi_x|^{\alpha}\, e^{J\psi_x^2}\, d\nu_{\omega}(\psi_x) \tag{3.1}$$

where $\alpha \in \mathbf{N}$ and J is the constant appearing in (1.5).

Inserting (1.4) in (2.4), we first write explicitly the function $q_x(\psi_x)$ as

$$q_x(\psi_x) = P_k(\psi_x) + \sum_{i=1}^{k-1} u_{2i} P_i(\psi_x) \tag{3.2}$$

where

$$P_i(\psi_x) = (\psi_x + \zeta_x)^{2i} - \zeta_x^{2i} - 2i\zeta_x^{2i-1}\psi_x = \sum_{j=2}^{2i} \binom{2i}{j} \zeta_x^{2i-j} \psi_x^j \tag{3.3}$$

Hence $q_x(\psi_x)$ is a polynomial of degree $2k$ in ψ_x without constant and linear term; namely it has the structure

$$q_x(\psi_x) = \sum_{i=2}^{2k} C_i(\zeta_x)\, \psi_x^i \tag{3.4}$$

where $C_i(\zeta_x)$ are also polynomials in ζ_x and $C_{2k}(\zeta_x) = 1$.

We now state the following lemma

Lemma 2. For any given $U(x)$ of the form (1.4) there exists a positive constant B_U, and two positive functions $C_U(J)$ and $F_U(J)$ satisfying, for some constants C, F

$$1 \leqslant C_U(J) \leqslant CJ^{\frac{1}{2k-2}} \qquad 1 \leqslant F_U(J) \leqslant FJ^{2k}$$

such that

(i) For $|\zeta_x| > C_U(J)$

$$q_x(\psi_x) - J\psi_x^2 \geqslant \tfrac{1}{4}\zeta_x^{2k-2}\psi_x^2 \tag{3.5}$$

$$q_x(\psi_x) \leqslant B_U \zeta_x^{2k-2}\psi_x^2 \qquad \text{whenever} \quad |\psi_x| \leqslant |\zeta_x| \tag{3.6}$$

(ii) For $|\zeta_x| \leqslant C_U(J)$

$$q_x(\psi_x) - J\psi_x^2 \geqslant \tfrac{1}{2}\psi_x^{2k} - F_U(J) \tag{3.7}$$

$$q_x(\psi_x) \leqslant 2\psi_x^{2k} + F_U(J) \tag{3.8}$$

The proof of Lemma 2 is given in the Appendix, together with explicit expressions for B_U, $C_U(J)$ and $F_U(J)$, see (A.11), (A.10) and (A.14). As a simple corollary of Lemma 2, we can now state the following lemma, which states the control of the form of (0.3) of the quantites $C_\alpha(J)$ defined in introduction.

Lemma 3. For any $\alpha \in \mathbf{N}$ and for all x

$$\int |\psi_x|^\alpha\, e^{J\psi_x^2}\, dv_\omega(\psi_x) \leqslant \Gamma\left(\frac{\alpha}{2}\right) 2^{\frac{\alpha}{2}} C(J, \zeta) \tag{3.9}$$

with

$$C(J, \zeta) = \begin{cases} \dfrac{C_k}{\zeta^{\alpha(k-1)}} & \text{if} \quad \zeta > C_U(J) \\ C_k\, e^{2F_U(J)} & \text{if} \quad \zeta \leqslant C_U(J) \end{cases} \tag{3.10}$$

where C_k and A are constants independent on J and ζ.

Note that, for fixed ζ, and for J large, $C(J,\zeta)$ increases as $C^{J^{2k}}$ for some C greater than 1, while $\lim_{\zeta\to\infty} C(J,\zeta)=0$ for any fixed J.

Proof. Consider first $|\zeta_x|\leqslant C_U(J)$. Recalling definition (2.8) and using (3.7) and (3.8) we get

$$\int |\psi_x|^\alpha e^{J\psi_x^2}\, d\nu_\omega(\psi_x)=\frac{\int_{\mathbf{R}} |\psi_x|^\alpha e^{-\{q_x(\psi_x)-J\psi_x^2\}}\, d\psi_x}{\int_{\mathbf{R}} e^{-q_x(\psi_x)}\, d\psi_x}\leqslant e^{2F_U(J)}\frac{\int_{\mathbf{R}} |\psi_x|^\alpha e^{-\psi_x^{2k}/2}\, d\psi_x}{\int_{\mathbf{R}} e^{-2\psi_x^{2k}}\, d\psi_x}$$

$$\leqslant e^{2F_U(J)} 2^{\frac{\alpha+2}{2k}}\Gamma\left(\frac{\alpha+1}{2k}\right)\left[\Gamma\left(\frac{1}{2k}\right)\right]^{-1}$$

$$\leqslant \Gamma\left(\frac{\alpha}{2}\right) A^{\frac{\alpha}{2}}\, C_{1,k}\, e^{2F_U(J)} \tag{3.11}$$

Then we consider the case $|\zeta_x|>C_U(J)$. Using that $C_U(J)\geqslant 1$ and $B_U\geqslant 1$ (see (A.11)), we have by (3.6)

$$\int d\psi_x\, e^{-q_x(\psi_x)}\geqslant 2\int_0^{|\zeta_x|} d\psi_x\, e^{-B_U\zeta_x^{2k-2}\psi_x^2}\geqslant 2\,\frac{1}{|\zeta_x|^{k-1}}\int_0^1 e^{-t^2}dt\geqslant \frac{1}{|\zeta_x|^{k-1}}$$

and by (3.5)

$$\int d\psi_x\, e^{-q_x(\psi_x)}\, |\psi_x|^\alpha\, e^{J\psi_x^2}\leqslant \int d\psi_x\, e^{-\frac{k}{2}\zeta_x^{2k-2}\psi_x^2}\, |\psi_x|^\alpha$$

$$=\Gamma\left(\frac{\alpha+1}{2}\right)\left[\frac{2}{k}\right]^{(\alpha+1)/2}\frac{1}{|\zeta_x|^{(1+\alpha)(k-1)}}$$

thus we obtain

$$\int |\psi_x|^\alpha e^{J\psi_x^2}\, d\nu_\omega(\psi_x)\leqslant \Gamma\left(\frac{\alpha+1}{2}\right)\left[\frac{2}{k}\right]^{(\alpha+1)/2}\frac{1}{|\zeta_x|^{\alpha(k-1)}}\leqslant \Gamma\left(\frac{\alpha}{2}\right)\frac{C_{2,k}}{|\zeta_x|^{\alpha(k-1)}} \tag{3.12}$$

Collecting (3.11) and (3.12) and putting $C_k=\max\{C_{1,k}, C_{2,k}\}$, we get

$$\int |\psi_x|^\alpha e^{J\psi_x^2}\, d\nu_\omega(\psi_x)\leqslant \Gamma\left(\frac{\alpha}{2}\right) A^{\frac{\alpha}{2}} C(J,|\zeta_x|)$$

Recalling that $\zeta = \inf_x |\zeta_x|$ and observing that $C(J, |\zeta_x|)$ is a decreasing function of $|\zeta_x|$ for any fixed J we have that $C(J, |\zeta_x|) \leqslant C(J, \zeta)$ for all x and J, which completes the proof.

4. POLYMER EXPANSION

Let us first recall some basic definitions about graphs in finite sets. In general, if A is any finite set, we denote by $|A|$ the number of elements of A. Given a finite set A, we define a *graph* g in A as a collection $\{\lambda_1, \lambda_2,..., \lambda_m\}$ of distinct pairs of A, i.e., $\lambda_i = \{x_i, y_i\} \subset R$ with $x_i \neq y_i$. The pairs $\lambda_1, \lambda_2,..., \lambda_m$ are called *links* of the graph g. We denote by $|g|$ the number of links in g. Given two graphs g and f we say that $f \subset g$ if each link of f is also a link of g.

A graph $g = \{\lambda_1, \lambda_2,..., \lambda_m\}$ in A is *connected* if for any pair B, C of subsets of A such that $B \cup C = A$ and $B \cap C = \emptyset$, there is a $\lambda_i \in g$ such that $\lambda_i \cap B \neq \emptyset$ and $\lambda_i \cap C \neq \emptyset$. If g is connected, then necessarely $\bigcup_{i=1}^m \lambda_i = A$ and $|A| - 1 \leqslant m \leqslant |A|(|A| - 1)/2$.

If g is a graph on A, then the elements of A are called *vertices* of g. We denote by G_A the set of all connected graphs in A.

A *tree* graph τ on $\{1,..., n\}$ is a connected graph such that $|\tau| = n - 1$. The set of all the tree graph over $\{1,..., n\}$ will be denoted by T_n. The *number of incidence* d_i of the vertex i of a tree graph $\tau \in T_n$ is the number of links $\lambda \in \tau$ such that $i \in \lambda$. We recall that for any τ and for any $i \in \tau$, the incidence numbers have the following properties: $1 \leqslant d_i \leqslant n - 1$ and $\sum_{i=1}^n d_i = 2n - 2$.

We now rewrite the "shifted" partition function $\tilde{Z}(\Lambda, \omega)$ and its logarithm (via Mayer expansion on the factor $e^{+\sum_{\{x, y\} \subset \Lambda} J_{xy}\psi_x\psi_y}$) in term of an hard core polymer gas. As it is well known we get

$$\tilde{Z}(\Lambda, \omega) = 1 + \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{R_1,..., R_n \subset \Lambda \\ R_i \cap R_j = \emptyset \ |R_i| \geqslant 2}} \rho(R_1) \cdots \rho(R_n) \tag{4.1}$$

where $R_1,..., R_n \subset \Lambda$ is a collection of subsets of Λ, called *polymers*, with activities $\rho(R)$ given by

$$\rho(R) = \int d\nu_\omega(\psi_R) \sum_{g \in G_R} \prod_{\{x, y\} \subset \Lambda} (e^{J_{xy}\psi_x\psi_y} - 1) \tag{4.2}$$

where $\int d\nu_\omega(\psi_R) = \int \prod_{x \in R} d\nu_\omega(\psi_x)$ and $\sum_{g \in G_R}$ is the sum over the connected graphs on the set R.

One has also

$$\log \tilde{Z}(\Lambda, \omega) = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{R_1,\ldots, R_n \subset \Lambda \\ |R_i| \geqslant 2}} \phi^T(R_1,\ldots, R_n)\, \rho(R_1) \cdots \rho(R_n) \tag{4.3}$$

with

$$\phi^T(R_1,\ldots, R_n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ \sum_{\substack{f \in G_n \\ f \subset g(R_1,\ldots, R_n)}} (-1)^{|f|} & \text{if} \quad n \geqslant 2 \text{ and } g(R_1,\ldots, R_n) \in G_n \end{cases} \tag{4.4}$$

where we denote by G_n the set of the connected graphs on $\{1,\ldots, n\}$ and by $g(R_1,\ldots, R_n)$ the graph in $\{1, 2,\ldots, n\}$ which has the link $\{i, j\}$ if and only if $R_i \cap Rj \neq \emptyset$.

Note that, if $g(R_1,\ldots, R_n)$ is not connected, then $\phi^T(R_1,\ldots, R_n) = 0$, since the sum on f in (4.4) runs over connected subgraphs of $g(R_1,\ldots, R_n)$.

The convergence of the expansion (4.3) is an argument widely studied by cluster expansion techniques. We shall see in Section 6 that the smallness of the quantity $JC(J, \zeta)$, where $C(J, \zeta)$ is the constant appearing in the estimate (3.9), is the basic tool needed to obtain such convergence. However, since such convergence is a byproduct of the convergence of the spin–spin truncated correlations (1.7), we give here their explicit expression in term of polymers,[18] and we treat directly the problem of the convergence of the correlations. This is achieved just recalling Lemma 1 and noting that the following identity holds

$$\tilde{\mu}^{\omega}_{\Lambda}(\psi_{x_1}, \psi_{x_2}) = \frac{\partial^k}{\partial \alpha_1 \partial \alpha_2} \log \tilde{Z}(\Lambda, \omega, \alpha_1, \alpha_2)|_{\alpha = 0} \tag{4.5}$$

where

$$\tilde{Z}(\Lambda, \omega, \alpha_1, \alpha_2) = \int d\nu_{\omega}(\psi_\Lambda)\, e^{+\sum_{\{x, y\} \subset \Lambda} J_{xy} \psi_x \psi_y} (1 + \alpha_1 \psi_{x_1})(1 + \alpha_2 \psi_{x_2})$$

It is now easy to expand $\tilde{Z}(\Lambda, \omega, \alpha_1, \alpha_2)$ in terms of polymers. For any $R \subset \Lambda$ let us denote by I_R the subset (possibly empty) of $\{1, 2\}$ such that $i \in I_R$ iff $x_i \in R$. We get

$$\tilde{Z}(\Lambda, \omega, \alpha_1, \alpha_2) = 1 + \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{R_1,\ldots, R_n \subset \Lambda \\ R_i \cap R_j = \emptyset \ |R_i| \geqslant 1}} \tilde{\rho}(R_1, \alpha) \cdots \tilde{\rho}(R_n, \alpha) \tag{4.6}$$

where

$$\tilde{\rho}(R, \alpha) = \begin{cases} \int d\nu_\omega(\psi_R) \prod_{i \in I_R} (1+\alpha_i \psi_{x_i}) \sum_{g \in G_R} \prod_{\{x, y\} \in g} (e^{J_{xy}\psi_x\psi_y} - 1) \\ \qquad\qquad\qquad\qquad\qquad \text{for} \quad |R| \geqslant 2 \\ \int d\nu_\omega(\psi_R) \prod_{i \in I_R} \alpha_i \psi_{x_i} \qquad \text{for} \quad I_R \neq \emptyset, \ |R| = 1 \\ 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad I_R = \emptyset, \ |R| = 1 \end{cases}$$

Note that also one-body polymers $R = \{x\}$ can contribute to the partition function (4.6), but only if $x = x_i$ for some $i \in \{1, 2\}$.

Now taking the log of (4.6) and observing, by (4.5), that only the terms proportional to $\alpha_1 \alpha_2$ will give a non vanishing contribution to the 2-points truncated correlation functions, we get

$$\begin{aligned} \mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2}) &= \tilde{\mu}_\Lambda^\omega(\psi_{x_1}, \psi_{x_2}) \\ &= \sum_{n \geqslant 1} \frac{1}{n!} \sum_{i_1, i_2 = 1}^{n} \sum_{\substack{R_1, \ldots, R_n \subset \Lambda, |R_j| \geqslant 2 \\ R_{i_1} \ni x_1 R_{i_2} \ni x_2}} \phi^T(R_1, \ldots, R_n) \, \tilde{\rho}(R_1) \cdots \tilde{\rho}(R_n) \end{aligned} \tag{4.7}$$

where

$$\tilde{\rho}(R_i) = \int d\nu_\omega(\psi_{R_i})(\psi_{x_1}^{\beta_i^1} + \beta_i^1 g(x_1))(\psi_{x_2}^{\beta_i^2} + \beta_i^2 g(x_2)) \sum_{g \in G_{R_i}} \prod_{\{x, y\} \in g} (e^{J_{xy}\psi_x\psi_y} - 1) \tag{4.8}$$

with

$$\beta_i^j = \begin{cases} 0 & \text{if} \quad i \neq i_j \\ 1 & \text{if} \quad i = i_j \end{cases} \tag{4.9}$$

and

$$g(x) = \int d\nu_\omega(\psi_x) \, \psi_x \tag{4.10}$$

Note that the one-body polymers are absorbed in the activity of the many body polymers (in the terms proportional to g), due to the fact that $R_1, \ldots, R_n$ must be connected and therefore each 1-body polymer (if any) is always contained in at least one many-body polymer. Remark also that, for any $x \in \Lambda$ we have

$$|g(x)| \leqslant \Gamma\left(\frac{1}{2}\right) C(0, \zeta) \tag{4.11}$$

5. THE DECAY OF THE SPIN–SPIN CORRELATIONS

In this section we state the main result of the paper and we make some remarks about its applicability.

Theorem. The spin–spin truncated correlation written as the series in the r.h.s. of (4.7) converges absolutely, uniformly in the configuration ω of the ω_x's defined in (1.6) and Λ, for $JC(J, \zeta)$ sufficiently small, where $C(J, \zeta)$ is the constant appearing in (3.9). Moreover it satisfies the following bounds.

(i) If J_{xy} is *finite range*, i.e., if there exists $\mathscr{R}$ such that $J_{xy} = 0$ if $|x-y| > \mathscr{R}$, then

$$|\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \leqslant C e^{-m(J, \zeta)\,|x_1 - x_2|} \tag{5.1}$$

where C is a constant uniform in ζ and Λ, and the "mass" $m(J, \zeta)$ which controls the exponential decay is bounded from below uniformly in ζ and Λ (i.e., $m(J, \zeta) \geqslant m(J)$ for all ζ, Λ); moreover for small J $m(J, \zeta) = O(\frac{1}{\mathscr{R}}\,|\log J|)$ and for large ζ $m(J, \zeta) = O(\frac{1}{\mathscr{R}}\log \zeta)$, i.e., the decay is stronger if the temperature J^{-1} or the external boundary parameter ζ are higher.

(ii) If $J_{xy} \leqslant B e^{-\gamma\,|x-y|}$

$$|\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \leqslant C' e^{-\frac{\gamma}{2}\,|x_1 - x_2|} \tag{5.2}$$

where C' is a constant uniform in ζ and Λ

(iii) If $\frac{C_1 J}{|x-y|^a} \leqslant |J_{xy}| \leqslant \frac{C_2 J}{|x-y|^a}$ with $a > d$

$$\frac{C_1' J}{|x_1 - x_2|^a} \leqslant |\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \leqslant \frac{C_2'' J}{|x_1 - x_2|^a} \tag{5.3}$$

where C_1', C_2'' are constants uniform in ζ and Λ.

Remarks.

(1) The result (i) cover the analogous result found in recent literature (see, e.g., refs. 3–8, 19). With our techniques however we need, to control the convergence of the expansions, the smallness of the quantity $JC(J, \zeta)$. Since $C(J, \zeta)$ is small for any J when $|\zeta|$ is large enough (see the first line of (3.10)) our theorem holds also for large couplings J_{xy} and ζ large depending on J. This allows us to cover some particular cases that can be meaningful

from a physical point of view and are not treated in the former related works. For example it is easy to see that when h_x has a definite sign on all the lattice sites and is large enough, and the boundary conditions have the same sign of h_x then $|\zeta|$ is large.

From a physical point of view this example is quite clear: even when the system is expected to have more than one phase, suitable external fields may force it in one phase. The same result may be obtained by suitable boundary conditions if they increase very rapidly with the volume. Note however that, in order to be more quantitative, the conditions on h and/or on the boundary condition have to take in account the details of the local interaction $U(\phi_x)$. As an example let us consider the case $U(x) = x^4$, $h_x = h$ and $J_{xy} > 0$ with J large. It is clear that, at least in the bulk, $\zeta_x \approx \zeta$ for all x, where z is the solution of $h + J\zeta = 4\zeta^3$. Then $|\zeta|$ is large when $|h|$ is large enough.

(2) The results (ii) and (iii) are not contained, as far as we know, in the previous literature. Recently Spohn and Zwerger[20] proved a result similar to (iii) in the particular case of the one-dimensional $O(N)$ spin model. They prove such result in a very different context, assuming a decay of the correlation and proving then that the decay is the same of the interaction. On the other hand they are able to obtain such result for any temperature above the critical one.

In case (ii) the equivalence between (5.2) and the log-Sobolev inequality is not yet proved. In case (iii) one should expect a slow decay of the dynamics.

(3) It is easy to prove in our framework the equivalence between the decay of the spin–spin correlations and the decay of the correlations in the form suggested by the Dobrushin–Shlosman condition in ref. 21 (see, e.g., refs. 3 and 7). In particular we are able to prove that

$$|\mu_\Lambda^\omega(f, g)| \leqslant C\, |||f|||\, |||g|||\, h(d(S_f, S_g)) \tag{5.4}$$

where S_u is the support of u,

$$|||u||| \equiv \sum_{x \in S_u} \sup_{\phi} \left| \frac{\partial}{\partial \phi_x} u(\phi) \right|$$

$d(S_f, S_g)$ is the minimal distance between the support of f, g and $h(r) = e^{-m(J,\zeta)\, r}$ in the case (i), $h(r) = e^{-\frac{\gamma}{2} r}$ in the case (ii) and $h(r) = r^{-a}$ in the case (iii). This equivalence can be proved as follows: one has easily

$$\mu_\Lambda^\omega(f, g) = \tilde{\mu}_\Lambda^\omega(f(\psi_{S_f} + \zeta_{S_f}) - f(\zeta_{S_f}), g(\psi_{S_g} + \zeta_{S_g}) - g(\zeta_{S_g}))$$

By the same argument leading to (4.7) one can obtain

$$\mu^{\omega}_{\Lambda}(f, g) = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{i_1, i_2 = 1}^{n} \sum_{\substack{R_1, \ldots, R_n \subset \Lambda, |R_j| \geqslant 2 \\ R_{i_1} \supset S_f, R_{i_2} \supset S_g}} \phi^T(R_1, \ldots, R_n)\, \tilde{\rho}(R_1) \cdots \tilde{\rho}(R_n)$$

where

$$\tilde{\rho}(R_i) = \int dv_{\omega}(\psi_{R_i}) [\beta_1^i [f(\psi_{S_f} + \zeta_{S_f}) - f(\zeta_{S_f})]\, \beta_2^i [g(\psi_{S_g} + \zeta_{S_g}) - g(\zeta_{S_g})]]$$

$$\sum_{\substack{R' \subseteq R_i \\ R' \supset R_i \backslash S_i}} \sum_{g \in G_{R'}} \prod_{\{x, y\} \in g} (e^{J_{xy} \psi_x \psi_y} - 1)$$

$$= \int dv_{\omega}(\psi_{R_i}) \left(\beta_1^i \sum_{x \in S_f} \psi_x \frac{\partial}{\partial \phi_x} f(\phi) \Big|_{\phi = \zeta + \tilde{\psi}_x} \right) \left(\beta_2^i \sum_{y \in S_g} \psi_y \frac{\partial}{\partial \phi_y} g(\phi) \Big|_{\phi = \zeta + \tilde{\psi}_y} \right)$$

$$\sum_{\substack{R' \subseteq R_i \\ R' \supset R_i \backslash S_i}} \sum_{g \in G_{R'}} \prod_{\{x, y\} \in g} (e^{J_{xy} \psi_x \psi_y} - 1)$$

where: $\zeta_x \leqslant \tilde{\psi}_x \leqslant \zeta_x + \psi_x$, $S_i = \emptyset$ if $i \neq i_1, i_2$; if $i_1 \neq i_2$ then $S_{i_1} = S_f$ $S_{i_2} = S_g$; if $i_1 = i_2 = i$ then $S_i = S_f \cup S_g$.

Then extracting the sup of the f and g derivatives and proceeding as in the proof of (5.1) one obtains (5.4).

(4) One can prove very easily the following

(iv) The free energy of the system at finite volume can be written as

$$F(\Lambda, \omega) = |\Lambda|^{-1} [C_{\omega}(\Lambda) + \log \tilde{Z}(\Lambda, \omega)] \tag{5.5}$$

where $C_{\omega}(\Lambda)$ is defined by (2.13) and $|\Lambda|^{-1} C_{\omega}(\Lambda)$ is uniform in the volume (but it diverges as $\zeta \to \infty$) while $|\Lambda|^{-1} \log \tilde{Z}(\Lambda, \omega)$ is the series in the r.h.s. of (4.3) which is analytic in $JC(J, \zeta)$ in a circle around the origin with radius independent on $|\Lambda|$.

Note that (iv) can be stated in terms of analyticity in J, since for bounded J, say $J < 1$, $C(J, \zeta) < C$ where C depends only on the form of U, and this gives an analyticity circle for J. See Appendix for more details.

(5) Our local interaction $U(\phi_x)$ is polynomial. One may try to generalize the theorem to more general U growing sufficiently fast to infinity. Althought it is clear that the shifted measure can be estimated from

above and below finding some analogous of Lemma 2, we did not find easily the way to have for a larger class of interactions a detailed control of the constants involved in the estimates. The regularizing effect of the large ζ should be preserved for potential U growing to infinity faster than quadratically.

6. PROOF OF THE THEOREM

Since the result (iii) is the more difficult, we discuss it in details. The proofs of (i), (ii) and (iv) can be obtained repeating the argument leading to (iii), or even in a simpler way. In the end of the section we present the sketchy argument giving the explicit behaviour of the rate of the exponential decay in the case (i) claimed in the theorem.

Proof of (iii). We will denote throughout the proof below with $O(1)$ any generic constant which depends only on a and d. The constant may change from line to line.

We observe first that the function $\tilde{\rho}(R)$ in (4.7), which specifies the activity of a polymer (4.8) depends on the polymer R also via the index $i \in I_R$, implying that in (4.7) one has to perform a sum over the two special indices i_1 and i_2 which are selected by the operator $\{\partial^2/\partial\alpha_1\,\partial\alpha_2\}|_{\alpha_1,\alpha_2=0}$. Hence we rewrite the sum (4.7) in the following more convenient way

$$\mu^{\omega}_{\Lambda}(\phi_{x_1}, \phi_{x_2}) = A_1(x_1, x_2) + A_2(x_1, x_2) \tag{6.1}$$

where

$$A_1(x_1, x_2) = \sum_{n \geqslant 2} \frac{1}{(n-2)!} \sum_{R_1 \ni x_1} \sum_{R_2 \ni x_2} \sum_{\substack{R_3,\ldots, R_n \subset \Lambda, \\ |R_j| \geqslant 2}} \phi^T(R_1,\ldots, R_n)\, \tilde{\rho}_1(R_1)\, \tilde{\rho}_2(R_2)\, \rho(R_3) \cdots \rho(R_n) \tag{6.2}$$

and

$$A_2(x_1, x_2) = \sum_{n \geqslant 1} \frac{1}{(n-1)!} \sum_{R_1 \supset \{x_1, x_2\}} \times \sum_{\substack{R_2,\ldots, R_n \subset \Lambda, \\ |R_j| \geqslant 2}} \phi^T(R_1,\ldots, R_n)\, \tilde{\rho}(R_{12})\, \rho(R_2) \cdots \rho(R_n) \tag{6.3}$$

where $\rho(R)$ is defined in (4.2), while

$$\tilde{\rho}_1(R)=\int d\nu_\omega(\psi_R)(\psi_{x_1}+g(x_1))\sum_{g\in G_R}\prod_{\{x,y\}\in g}(e^{J_{xy}\psi_x\psi_y}-1) \tag{6.4}$$

$$\tilde{\rho}_2(R)=\int d\nu_\omega(\psi_R)(\psi_{x_2}+g(x_2))\sum_{g\in G_R}\prod_{\{x,y\}\in g}(e^{J_{xy}\psi_x\psi_y}-1) \tag{6.5}$$

$$\tilde{\rho}_{12}(R)=\int d\nu_\omega(\psi_R)(\psi_{x_1}+g(x_1))(\psi_{x_2}+g(x_2))\sum_{g\in G_R}\prod_{\{x,y\}\in g}(e^{J_{xy}\psi_x\psi_y}-1) \tag{6.6}$$

In what follows we will denote with $\tilde{\rho}(R)$ an activity that can be $\tilde{\rho}_i(R)$, $\tilde{\rho}_{12}(R)$ or $\rho(R)$

We will need the following lemma

Lemma 4. If $JC(J,\zeta)$ is sufficiently small, it exists a positive function $\varepsilon(J,\zeta)$ such that, for any $z\in\Lambda$, $z'\in\Lambda$ with $z\neq z'$

$$\sum_{\substack{R:\,|R|\geqslant 2\\ z,z'\in R}}|\tilde{\rho}(R)|\,e^{|R|}\leqslant\frac{\varepsilon(J,\zeta)}{|z-z'|^a} \tag{6.7}$$

with

$$\varepsilon(J,\zeta)=O(1)\,C^2(J,\zeta)\,J$$

Note that from Lemma 4 it is immediate to obtain

Corollary 5.

$$\sup_{x\in\mathbf{Z}^{\mathbf{d}}}\sum_{R:\,x\in R}|\tilde{\rho}(R)|\,e^{|R|}\leqslant O(1)\,\varepsilon(J,\zeta) \tag{6.8}$$

The basic tool to prove Lemma 4 and Corollary 5 is the *Brydges–Battle–Federbush tree graph inequality*, namely the following lemma.

Lemma 6. Let V_{ij}, $1\leqslant i<j\leqslant n$ be a set of real numbers and V_{ii} ($i=1,2,\ldots,n$) be positive numbers such that, for any subset $S\subset\{1,2,\ldots,n\}$

$$\sum_{i\in S}V_{ii}+\sum_{\{i,j\}\in S}V_{ij}\geqslant 0$$

Then

$$\left|\sum_{g\in G_n}\prod_{\{i,j\}\in g}(e^{-V_{ij}}-1)\right|\leqslant e^{\sum_{i=1}^{n}V_{ii}}\sum_{\tau\in T_n}\prod_{\{i,j\}\in\tau}|V_{ij}| \tag{6.9}$$

We recall that G_n denotes the set of the connected graphs on $\{1,2,\ldots,n\}$ and T_n denotes the set of the tree graphs on $\{1,2,\ldots,n\}$. For the proof of this lemma see, e.g., refs. 22–24.

Proof of Lemma 4. For simplicity we bound $|\tilde{\rho}(R)|$ in the case in which $R\cap\{x_1,x_2\}=\emptyset$ so that $\tilde{\rho}=\rho$ as defined in (4.2). The other cases are treated analogously. By (6.9), and observing that for any R,

$$\sum_{x,y\in R}J_{xy}\psi_x\psi_y\leqslant\sum_{x\in R}J\psi_x^2$$

Lemma 6 can be used with $V_{ij}\equiv -J_{xy}\psi_x\psi_y$ and $V_{ii}\equiv J\psi_x^2$, obtaining

$$|\rho(R)|\leqslant\left(\prod_{x\in R}\int dv_\omega(\psi_x)\right)e^{J\sum_{x\in R}\psi_x^2}\sum_{\tau\in T_R}\prod_{\{x,y\}\in g}|\psi_x|\,|\psi_y|\,|J_{xy}| \tag{6.10}$$

then

$$\begin{aligned}
&\sum_{\substack{R\subset\Lambda:\,|R|\geqslant 2\\ z,z'\in R}}|\rho(R)|\,e^{|R|}\\
&\quad\leqslant\sum_{n\geqslant 3}e^n\sum_{\substack{R\subset\Lambda,\,z,z'\in R\\ |R|=n}}|\rho(R)|\\
&\quad\leqslant\sum_{n\geqslant 2}\frac{e^n}{(n-2)!}\sum_{\substack{x_3,\ldots,x_n\\ x_i\in\Lambda,\,x_i\neq x_j,\,\forall i,j\\ x_1=z,\,x_2=z'}}\int\prod_{i=1}^{n}dv_\omega(\psi_{x_i})\,e^{J\psi_{x_i}^2}\sum_{\tau\in T_n}\prod_{\{i,j\}\in\tau}|\psi_{x_i}|\,|\psi_{x_j}|\,|J_{x_ix_j}|\\
&\quad\leqslant\sum_{n\geqslant 2}\frac{e^n}{(n-2)!}\sum_{\substack{x_3,\ldots,x_n\\ x_i\in\Lambda,\,x_i\neq x_j,\,\forall i,j\\ x_1=z,\,x_2=z'}}\sum_{\tau\in T_n}\int\prod_{i=1}^{n}dv_\omega(\psi_{x_i})\,e^{J\psi_{x_i}^2}\,|\psi_{x_i}|^{d_i}\prod_{\{i,j\}\in\tau}|J_{x_ix_j}|\\
&\quad\leqslant\sum_{n\geqslant 2}\frac{[AeC(J,\zeta)]^n}{(n-2)!}\sum_{\tau\in T_n}\left\{\prod_{i=1}^{n}\Gamma\left(\frac{d_i}{2}\right)\sum_{\substack{x_3,\ldots,x_n\\ x_i\in\Lambda,\,x_i\neq x_j,\,\forall i,j\\ x_1=z,\,x_2=z'}}\prod_{\{i,j\}\in\tau}|J_{x_ix_j}|\right\}
\end{aligned}$$

Recall that, for a fixed $\tau \in T_n$, d_i is the incidence number of the vertex i, i.e., d_i is the number of links $\{j, k\} \in \tau$ such that $j = i$ or $k = i$. Note also that in the last line above we have used the bound (3.9) and the property $\sum_{i=1}^n d_i = 2n-2$. We now use the fact that for any τ in $\{1, 2,..., n\}$, there is a unique path $\bar{\tau}$ in τ which joins vertex 1 to vertex 2. Let us call $I_\tau \equiv \{1, i_1,..., i_k, 2\}$ the subset of $\{1, 2, 3,..., n\}$ whose elements are the vertices of the path $\bar{\tau}$. Note that this set is ordered, i.e., τ establishes uniquely the order of this set in the sense that the sub-tree $\bar{\tau}$ is given explicitly by the set of bonds $\bar{\tau} = \{1, i_1\}, \{i_1, i_2\}, \{i_2, i_3\},..., \{i_{k-1}, i_k\}\{i_k, 2\}$.

Then one can easily check that

$$\sum_{\substack{x_3,..., x_n \\ x_i \in \Lambda,\, x_i \neq x_j,\, \forall i, j \\ x_1 = z,\, x_2 = z'}} \prod_{\{i, j\} \in \tau} J_{x_i x_j}$$

$$\leqslant J^{d_1-1} J^{d_2-1} \prod_{i \notin I_\tau} J^{d_i-1} \prod_{\substack{i \in I_\tau \\ i \neq 1, 2}} J^{d_i-2} \sum_{\substack{x_{i_1},..., x_{i_k} \\ x_{i_j} \in \Lambda,\, x_{i_j} \neq x_{i_s},\, \forall i, j}} J_{x_1 x_{i_1}} J_{x_{i_1} x_{i_2}} \cdots J_{x_{i_k} x_2}$$

We have by definition that $J = O(1)\, J_0$. Moreover

$$\sum_{\substack{x_{i_1},..., x_{i_k} \\ x_{i_j} \in \Lambda,\, x_{i_j} \neq x_{i_s},\, \forall i, j}} J_{x_1 x_{i_1}} J_{x_{i_1} x_{i_2}} \cdots J_{x_{i_k} x_2}$$

$$= J_0^k \sum_{\substack{x_{i_1},..., x_{i_k} \\ x_{i_j} \in \Lambda,\, x_{i_j} \neq x_{i_s},\, \forall i, j}} \frac{1}{|x_1 - x_{i_1}|^a} \frac{1}{|x_{i_1} - x_{i_2}|^a} \cdots \frac{1}{|x_{i_k} - x_2|^a}$$

$$\leqslant \frac{J^k [O(1)]^k}{|x_1 - x_2|^a}$$

where the last line follows applying iteratively the inequality

$$\sum_{\substack{\bar{x} \in \Lambda \\ \bar{x} \neq x, y}} \frac{1}{|x - \bar{x}|^a} \frac{1}{|\bar{x} - y|^a} \leqslant O(1) \frac{1}{|x - y|^a}$$

Hence, recalling that for any tree τ we have $\sum_{i=1}^n (d_i - 1) = n-1$, we get

$$\sum_{\substack{x_3,..., x_n \\ x_i \in \Lambda,\, x_i \neq x_j,\, \forall i, j \\ x_1 = z,\, x_2 = z'}} \prod_{\{i, j\} \in \tau} J_{x_i x_j} \leqslant \frac{[J O(1)]^{n-1}}{|z - z'|^a}$$

Recalling now Cayley formula, i.e.,

$$\sum_{\substack{\tau \in T_n \\ d_1,\ldots, d_n \text{ fixed}}} 1 = \frac{(n-2)!}{\prod_{i=1}^{n} (d_i - 1)!}$$

we get

$$\sum_{R \subset \Lambda,\, z,\, z' \in R} |\rho(R)|\, e^{|R|}$$

$$\leqslant \sum_{n \geqslant 2} \frac{[AeC(J, \zeta)]^n}{(n-2)!} \frac{[JO(1)]^{n-1}}{|z - z'|^a} \sum_{\substack{d_1 + \cdots + d_n = 2n-2 \\ d_i \geqslant 1}} \left\{ \prod_{i=1}^{n} \Gamma\left(\frac{d_i}{2}\right) \frac{(n-2)!}{\prod_{i=1}^{n} (d_i - 1)!} \right\}$$

$$\leqslant \sum_{n \geqslant 2} [4AeC(J, \zeta)]^n [JO(1)]^{n-1} \frac{1}{|z - z'|^a} \leqslant \frac{O(1)\, C^2(J, \zeta)\, J}{|z - z'|^a}$$

provided $4AeO(1)\, C(J, \zeta)\, J < 1$. This proves Lemma 4.

Upper Bound for the Correlations

From (6.1) we can write

$$|\mu^{\omega}_{\Lambda}(\phi_{x_1}, \phi_{x_2})| \leqslant |A_1(x_1, x_2)| + |A_2(x_1, x_2)| \tag{6.11}$$

Let us thus now find an upper bound for the term $|A_1(x_1, x_2)|$.

$$|A_1(x_1, x_2)|$$

$$\leqslant \sum_{\substack{R_1, R_2 : |R_i| \geqslant 2 \\ x_1 \in R_1,\, x_2 \in R_2}} |\tilde{\rho}_1(R_1)|\, |\tilde{\rho}_2(R_2)|$$

$$\times \left[|\phi^T(R_1, R_2)| + \sum_{n \geqslant 3} \frac{1}{(n-2)!} \sum_{\substack{R_3,\ldots, R_n \subset \Lambda \\ |R_i| \geqslant 2}} |\phi^T(R_1, R_2,\ldots, R_n)\, \rho(R_3) \cdots \rho(R_n)| \right]$$

$$\leqslant \sum_{\substack{R_1, R_2 : |R_i| \geqslant 2 \\ x_1 \in R_1,\, x_2 \in R_2}} |\tilde{\rho}_1(R_1)|\, |\tilde{\rho}_2(R_2)|\, \phi^T(R_1, R_2) + \sum_{n \geqslant 3} \frac{1}{(n-2)!} B_n(x_1, x_2)$$

where

$$B_n(x_1, x_2)$$

$$= \sum_{\substack{R_1,\ldots, R_n \subset \Lambda \\ |R_i| \geqslant 2,\, x_1 \in R_1,\, x_2 \in R_2}} |\phi^T(R_1, R_2,\ldots, R_n)|\, |\tilde{\rho}_1(R_1)|\, |\tilde{\rho}_2(R_2)|\, |\rho(R_3) \cdots \rho(R_n)|$$

Now we can reorganize the sum over the sets $R_1,..., R_n$ using the fact that $\phi^T(R_1,..., R_n)$ depends only on the graph $g(R_1,..., R_n) \in G_n$. From the explicit definition (4.4) of $\phi^T(R_1,..., R_n)$ we obtain

$$B_n(x_1, x_2) = \sum_{g \in G_n} \left| \sum_{\substack{f \in G_n \\ \subset g}} (-1)^{|f|} \right| \times \sum_{\substack{R_1,..., R_n \subset \Lambda: |R_i| \geqslant 2 \\ g(R_1,..., R_n) = g,\, x_1 \in R_1,\, x_2 \in R_2}} |\tilde{\rho}_1(R_1)|\, |\tilde{\rho}_2(R_2)|\, |\rho(R_3) \cdots \rho(R_n)| \tag{6.12}$$

By the Rota formula we have

$$\left| \sum_{\substack{f \in G_n \\ f \subset g}} (-1)^{|f|} \right| \leqslant N(g) \tag{6.13}$$

where $N(g)$ denotes the number of connected tree graphs in g. The proof of the Rota formula above can be found, e.g., in ref. 18. See ref. 24 for a simpler proof using the Brydges–Battle–Federbush tree graph identity.[22, 23]

We observe now that

$$\sum_{g \in G_n} [\cdot] = \sum_{\tau \in T_n} \sum_{g:\, \tau \subset g} \frac{1}{N(g)} [\cdot] \tag{6.14}$$

Such equality can be proved as follows. First, we fix a connected tree graph τ in T_n, then we sum, for τ fixed, over all connected graphs in G_n which contain τ as a subgraph. We are clearly counting too much, since for the same connected graph g in G_n there are exactly $N(g)$ tree graphs which are contained in it. Thus in the double sum $\sum_\tau \sum_{g \supset t}$ each g will be repeated exacly $N(g)$ times. Whence the presence of the factor $1/N(g)$ to correct this double counting.

Inserting (6.13) and (6.14) in (6.12) we obtain

$$B_n(x_1, x_2) = \sum_{\tau \in T_n} w_\tau(x_1, x_2) \tag{6.15}$$

where we have defined

$$w_\tau(x_1, x_2) = \sum_{\substack{R_1,..., R_n \subset \Lambda: |R_i| \geqslant 2 \\ g(R_1, R_2,..., R_n) \supset \tau,\, x_1 \in R_1,\, x_2 \in R_2}} |\tilde{\rho}_1(R_1)|\, |\tilde{\rho}_2(R_2)|\, |\rho(R_3) \cdots \rho(R_n)|$$

Using now the obvious bound

$$\sum_{R:\, R \cap R' \neq \varnothing} [\,\cdot\,] \leqslant |R'| \sup_{x \in R'} \sum_{R:\, x \in R} [\,\cdot\,]$$

and calling again $\bar{\tau}$ the subtree of τ which is the unique path joining vertex 1 to vertex 2 and $I_\tau = \{1, i_1,..., i_k, 2\}$ the ordered set of the vertices of $\bar{\tau}$, then one can easily check that

$$\begin{aligned}
&w_\tau(x_1, x_2)\\
&\leqslant \prod_{i \notin I_\tau}^{n} \left[\sup_{x \in Z^d} \sum_{R_i:\, x \in R_i} |R_i|^{d_i - 1} |\rho(R_i)| \right]\\
&\times \sum_{\substack{R_1, R_{i_1},..., R_{i_k}, R_2 \\ R_1 \cap R_{i_1} \neq \varnothing,..., R_{i_k} \cap R_2 \neq \varnothing}} |R_1|^{d_1 - 1} |\tilde{\rho}_1(R_1)|\, |R_2|^{d_2 - 1} |\tilde{\rho}_2(R_2)| \prod_{\substack{i \in I_\tau \\ i \neq 1, 2}}^{n} |R_i|^{d_i - 2} |\rho(R_i)|\\
&\leqslant \prod_{i \notin I_\tau}^{n} \left[\sup_{x \in Z^d} \sum_{R_i:\, x \in R_i} (d_i - 1)!\, |\rho(R_i)|\, e^{|R_i|} \right] (d_1 - 1)!\, (d_2 - 1)!\\
&\quad \times \sum_{\substack{R_1, R_{i_1},..., R_{i_k}, R_2 \\ R_1 \cap R_{i_1} \neq \varnothing,..., R_{i_k} \cap R_2 \neq \varnothing}} |\tilde{\rho}_1(R_1)|\, e^{|R_1|}\, |\tilde{\rho}_2(R_2)|\, e^{|R_2|} \prod_{\substack{i \in I_\tau \\ i \neq 1, 2}} (d_i - 2)!\, |\rho(R_i)|\, e^{|R_i|}
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\sum_{\substack{R_1, R_{i_1},..., R_{i_k}, R_2:\, x_1 \in R_1,\, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \varnothing,..., R_{i_k} \cap R_2 \neq \varnothing}}\\
&\leqslant \sum_{x_{i_0} \in Z^d} \sum_{x_{i_1} \in Z^d} \cdots \sum_{x_{i_k} \in Z^d} \sum_{\substack{R_1 \\ x_1, x_{i_0} \in R_1}} \sum_{\substack{R_{i_1} \\ x_{i_0}, x_{i_1} \in R_{i_1}}} \sum_{\substack{R_{i_2} \\ x_{i_1}, x_{i_2} \in R_{i_2}}} \cdots \sum_{\substack{R_{i_k} \\ x_{i_{k-1}}, x_{i_k} \in R_{i_k}}} \sum_{\substack{R_2 \\ x_{i_k}, x_2 \in R_2}}
\end{aligned}$$

and hence recalling (6.7)

$$\begin{aligned}
&\sum_{\substack{R_1, R_{i_1},..., R_{i_k}, R_2:\, x_1 \in R_1,\, x_2 \in R_2 \\ R_1 \cap R_{i_1} \neq \varnothing,..., R_{i_k} \cap R_2 \neq \varnothing}} |\tilde{\rho}_1(R_1)|\, e^{|R_1|}\, |\tilde{\rho}_2(R_2)|\, e^{|R_2|} \prod_{i \in I_\tau} |\rho(R_i)|\, e^{|R_i|}\\
&\leqslant \sum_{x_{i_0} \in Z^d} \sum_{x_{i_1} \in Z^d} \cdots \sum_{x_{i_k} \in Z^d} [\varepsilon(J, \zeta)]^{k+2}\\
&\quad \times \frac{1}{|x_1 - x_{i_0}|^a} \frac{1}{|x_{i_0} - x_{i_1}|^a} \frac{1}{|x_{i_1} - x_{i_2}|^a} \cdots \frac{1}{|x_{i_{k-1}} - x_{i_k}|^a} \frac{1}{|x_{i_k} - x_2|^a}\\
&\leqslant [\varepsilon(J, \zeta)]^{k+2}\, O(1)^{k+2} \frac{1}{|x_1 - x_2|^a}
\end{aligned}$$

Thus we obtain, using also Corollary 5 and observing that $|\{1,..., n\}\setminus I_\tau| = n-k-2$

$$w_\tau(x_1, x_2) \leqslant (d_1-1)!\,(d_2-1)! \prod_{i\notin I_\tau}^{n} \left[\sup_{x\in Z^d} \sum_{R_i : x\in R_i} (d_i-1)!\,|\rho(R_i)|\, e^{|R_i|}\right]$$

$$\times \prod_{i\in I_\tau} (d_i-2)!\,[\varepsilon(J,\zeta)]^{k+2}\, O(1)^{k+2} \frac{1}{|x_1-x_2|^a}$$

$$\leqslant \prod_{i=1}^{n} (d_i-1)!\,[O(1)\,\varepsilon(J,\zeta)]^n \frac{1}{|x_1-x_2|^a}$$

Summing finally over τ (using once again Cayley formula) we obtain

$$B_n(x_1, x_2) \leqslant (n-2)!\,[O(1)\,\varepsilon(J,\zeta)]^n \frac{1}{|x_1-x_2|^a}$$

Thus, taking $C(J,\zeta)\,J$ such small to make $O(1)\,\varepsilon(J,\zeta)<1$, we get for the contribution A_1 to the correlations the following bound:

$$|A_1(x_1,x_2)| \leqslant \sum_{\substack{R_1, R_2 : |R_i|\geqslant 2\\ x_1\in R_1,\, x_2\in R_2}} |\tilde\rho_1(R_1)|\,|\tilde\rho_2(R_2)|\,|\phi^T(R_1,R_2)|$$

$$+\sum_{n\geqslant 3} [O(1)\,\varepsilon(J,\zeta)]^n \frac{1}{|x_1-x_2|^a}$$

$$\leqslant \sum_{x\in Z^d} \sum_{\substack{R_1 : |R_1|\geqslant 2\\ x_1, x\in R_1}} \sum_{\substack{R_2 : |R_2|\geqslant 2\\ x_2, x\in R_2}} |\tilde\rho_1(R_1)|\,|\tilde\rho_2(R_2)| + O(1)[\varepsilon(J,\zeta)]^3 \frac{1}{|x_1-x_2|^a}$$

$$\leqslant O(1)[\varepsilon(J,\zeta)]^2 \frac{1}{|x_1-x_2|^a} + O(1)[\varepsilon(J,\zeta)]^3 \frac{1}{|x_1-x_2|^a}$$

$$\leqslant O(1)[\varepsilon(J,\zeta)]^2 \frac{1}{|x_1-x_2|^a}$$

i.e., in conclusion, for $JC(J,\zeta)$ sufficiently small we can find a constant $A_1>0$ uniformly in ζ and Λ such that

$$|A_1(x_1,x_2)| \leqslant \frac{A_1(JC(J,\zeta))^2}{|x_1-x_2|^a} \tag{6.16}$$

In a similar and much easier way one can also prove an analogous bound on $|A_2(x_1, x_2)|$ of the form

$$|A_2(x_1, x_2)| \leqslant \frac{A_2 JC(J, \zeta)}{|x_1 - x_2|^a} \tag{6.17}$$

for $JC(J, \zeta)$ sufficiently small and for some constant $A_2 > 0$ uniform in ζ and Λ. Note that $|A_1(x_1, x_2)|$ and $|A_2(x_1, x_2)|$ are small quantities and $|A_1(x_1, x_2)|$ is of the order of $(JC(J, \zeta))^2$ while $|A_2(x_1, x_2)|$ is of the order of $JC(J, \zeta)$.

Hence by (6.11), (6.16) and (6.17) we get

$$|\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \leqslant O(1) \frac{JC(J, \zeta)}{|x_1 - x_2|^a}$$

for $JC(J, \zeta)$ sufficiently small.

Lower Bound for Correlations

Since we proved, by the above computations, that the correlations are analytic in the parameter $JC(J, \zeta)$, it is enough to prove that the lower order term in $JC(J, \zeta)$ decays as the upper bound.

Again by (6.1) we can write

$$\begin{aligned}|\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \geqslant \bigg| \int d\nu_\omega(\psi_{x_1}) \int d\nu_\omega(\psi_{x_2})(\psi_{x_1} + g(x_1))(\psi_{x_2} + g(x_2)) \\ \times (e^{J_{x_1 x_2} \psi_{x_1} \psi_{x_2}} - 1) \bigg| + O((JC(J, \zeta)^2)\end{aligned}$$

Moreover

$$\begin{aligned}&\left| \int d\nu_\omega(\psi_{x_1}) \int d\nu_\omega(\psi_{x_2})(\psi_{x_1} + g(x_1))(\psi_{x_2} + g(x_2))(e^{J_{x_1 x_2} \psi_{x_1} \psi_{x_2}} - 1) \right| \\ &\quad \geqslant |J_{x_1 x_2}| \left| \int d\nu_\omega(\psi_{x_1}) \int d\nu_\omega(\psi_{x_2})(\psi_{x_1} + g(x_1))\, \psi_{x_1} (\psi_{x_2} + g(x_2))\, \psi_{x_2} \right| \\ &\quad\quad + O((JC(J, \zeta)^2)\end{aligned}$$

and finally

$$|J_{x_1 x_2}| \left| \int dv_\omega(\psi_{x_1}) \int dv_\omega(\psi_{x_2})(\psi_{x_1} + g(x_1))\, \psi_{x_1} (\psi_{x_2} + g(x_2))\, \psi_{x_2} \right|$$
$$\geqslant O(1)\, \frac{JC(J, \zeta)}{|x_1 - x_2|^a}$$

In the last line we use a trivial generalization of Lemma 3.

The proofs of (i) and (ii) can be done along the same lines of the proof of (iii). Let us give here just a sketchy argument in order to show that the rate of the exponential decay of correlations in the case of finite range potential is indeed of the order $\frac{\log \varepsilon(J, \zeta)}{\mathscr{R}}$, where $\mathscr{R}$ is the range of the potential. Let $d(R) = \sup_{x, y \in R} |x - y|$. Observe that for each term of the series (4.7) the n-ple of polymers $R_1, \ldots, R_n$ has to be connected, thus in particular they must connect x_1 with x_2. Observe, by (4.8) that if $J_{xy} = 0$ for $|x - y| \geqslant \mathscr{R}$, then $\rho(R) = 0$ unless $|R| > \frac{d(R)}{\mathscr{R}}$. Moreover it is easy to check by using Lemma 4 that

$$\sup_x \sum_{\substack{R:\, x \in R \\ |R| \geqslant n}} |\rho(R)|\, e^{|R|} \leqslant O(1)\, \varepsilon(J, \zeta)^n$$

Hence, one can argue that the lower order term in $\varepsilon(J, \zeta)$ in the series (4.7) is

$$\varepsilon^{|x_1 - x_2|/\mathscr{R}}$$

Then for the finite range case we get

$$|\mu_\Lambda^\omega(\phi_{x_1}, \phi_{x_2})| \leqslant Ce^{-m(J, \omega)\, |x_1 - x_2|)}$$

with

$$m(J, \omega) = \frac{O(1)\, |\log[JC(J, \zeta)]|}{\mathscr{R}}$$

7. SOME OPEN QUESTIONS

The technique presented in this paper allowed us to find some results about the decay of the truncated correlations. Such results seem to be in some sense optimal, as it is shown by the lower bound of the decay of the correlations proved in our theorem, part (iii).

With this respect the overall constant in front of the free energy due to the possibly bad boundary conditions and the possibly large value of the simple expectations of the fields play actually no role.

A careful control of these two topics, which corresponds to a control of the dependence of the ζ_x's on the boundary condition, may be useful in order to prove the uniqueness of the Gibbs measure in the sense presented, e.g., in ref. 14. With our technique it is maybe possible to prove such unicity also in some new context, say large magnetic field and/or power decay of the interaction. The subject is under study.

APPENDIX. PROOF OF LEMMA 2

Proof of Part (i). Let us first prove the following inequalities, valid for all $i = 1, 2,\ldots, k$

$$P_i(\psi_x) \geqslant i\zeta_x^{2i-2}\psi_x^2 \tag{A.1}$$

$$P_i(\psi_x) \leqslant 4^i\zeta_x^{2i-2}\psi_x^2 \qquad \text{whenever} \quad |\psi_x| \leqslant |\zeta_x| \tag{A.2}$$

where $P_i(\psi_x)$ is defined in (3.3).

(A.2) follows elementary from the expression of $P_i(\psi_x)$

Concerning (A.1), putting $f_i(\psi_x) = P_i(\psi_x) - i\zeta_x^{2i-2}\psi_x^2$, we have to show that $f_i(\psi_x) \geqslant 0$ for all $i = 1,\ldots, k$. We find the minima and the maxima of $f_i(\psi_x)$. Consider just the case $i > 1$, since for $i = 1$ we have trivially $f_1(\psi_x) = 0$, hence $f_1(\psi_x) \geqslant 0$. Thus $f'_i(\psi_x) = 0 \Rightarrow 2i(\psi_x + \zeta_x)^{2i-1} - 2i\zeta_x^{2i-1} - 2i\zeta_x^{2i-2}\psi_x = 0 \Rightarrow (\psi_x + \zeta_x)^{2i-1} = \zeta_x^{2i-2}(\psi_x + \zeta_x)$. Hence the real solutions of $f'_i(\psi_x) = 0$ are $\psi_x = 0$, $\psi_x = -\zeta_x$ and $\psi_x = -2\zeta_x$. Moreover for any $i > 1$, we have $\lim_{\psi_x \to \imath} f_i(\psi_x) = +\imath$, and then $f_i(\psi_x = 0) = 0$ and $f_i(\psi_x = -2\zeta_x) = 0$ are two absolute minima for $f_i(\psi_x)$ and hence $f_i(\psi_x) \geqslant 0$.

We now prove the following inequality

$$P_i(\psi_x) \geqslant \tfrac{1}{2}\psi_x^{2i} \qquad \text{whenever} \quad |\psi_x| \geqslant |\zeta_x| \tag{A.3}$$

Observe first that both $\frac{1}{2}\psi_x^{2i}$ and $P_i(\psi_x)$ are convex functions of ψ_x with a minimum in $\psi_x = 0$. Moreover $P_i(\psi_x = \zeta_x) = (4^i - 2i - 1)\,\zeta_x^{2i} \geqslant \frac{1}{2}\zeta_x^{2i}$ and $P_i(\psi_x = -\zeta_x) = (2i-1)\,\zeta_x^{2i} \geqslant \frac{1}{2}\zeta_x^{2i}$. Thus $P_i(\psi_x) \geqslant \frac{1}{2}\psi_x^{2i}$ for $|\psi_x| \geqslant |\zeta_x|$ and for any $i = 1, 2,\ldots, k$.

We can now prove first part of the lemma, i.e., (3.5).

Suppose first that $|\psi_x| \leqslant |\zeta_x|$, thus we can use (A.1) (which is valid for any ψ_x) and obtain

$$q_x(\psi_x) - J\psi_x^2 \geqslant \frac{k}{2}\zeta_x^{2k-2}\psi_x^2 + \left[\frac{k}{2}\zeta_x^{2k-2} - \sum_{i=1}^{k-1} 4^i\,|u_{2i}|\,\zeta_x^{2i-2} - J\right]\psi_x^2$$

Put $F_U(z)=\frac{k}{2}z^{2k-2}-\sum_{i=1}^{k-1}4^i\,|u_{2i}|\,z^{2i-2}$. Clearly, since $k>1$, it exists $R>0$ such that $F_U(z)\geqslant\frac{1}{2}z^{2k-2}$ for all $|z|\geqslant R$. Define now

$$C_U^1=\inf\{R>0\text{: for all } |z|\geqslant R \quad F_U(z)\geqslant\tfrac{1}{2}z^{2k-2}\} \tag{A.4}$$

Thus for $|\zeta_x|\geqslant C_U^1$ we have that $F_U(\zeta_x)-J\geqslant\frac{1}{2}\zeta_x^{2k-2}-J$. If we now put

$$C_U^1(J)=\begin{cases} C_U^1 & \text{if} \quad J\leqslant\frac{1}{2}C_U^{2k-2} \\ \sqrt[2k-2]{2J} & \text{if} \quad J>\frac{1}{2}C_U^{2k-2} \end{cases} \tag{A.5}$$

we obtain that $F_U(\zeta_x)-J\geqslant 0$ for all $|\zeta_x|\geqslant C_U^1(J)$, or in other words

$$q_x(\psi_x)-J\psi_x^2\geqslant\frac{k}{2}\zeta_x^{2k-2}\psi_x^2 \quad \text{whenever } \psi_{\rm x}\in(-|\zeta_{\rm x}|,+|\zeta_{\rm x}|) \text{ and } |\zeta_x|\geqslant C_U^1(J) \tag{A.6}$$

Suppose now $|\psi_x|\geqslant|\zeta_x|$, then we can use (A.3) and obtain

$$q_x(\psi_x)-J\psi^2\geqslant\frac{1}{4}\psi_x^{2k}+\left[\frac{1}{4}\psi_x^{2k}-\sum_{i=1}^{k-1}4^i\,|u_{2i}|\,\psi_x^{2i}\right]-J\psi_x^2$$

Put $G_U(z)=\frac{1}{4}z^{2k}-\sum_{i=1}^{k-1}4^i\,|u_{2i}|\,z^{2i}$. Then it exists R' such that $G_U(z)>\frac{1}{5}z^{2k}$ for all $|z|>R'$. Define

$$C_U^2=\inf\{R'>0\text{: for all } |z|\geqslant R' \quad G_U(z)\geqslant\tfrac{1}{5}z^{2k}\} \tag{A.7}$$

Thus for $|\psi_x|\geqslant C_U^2$ we have that $G_U(\psi_x)-J\psi^2\geqslant\frac{1}{5}\psi_x^2[\psi_x^{2k-2}-5J]$. If we now put

$$C_U^2(J)=\begin{cases} C_U^2 & \text{if} \quad J\leqslant\frac{1}{2}C_U^{2k-2} \\ \sqrt[2k-2]{5J} & \text{if} \quad J>\frac{1}{5}C_U^{2k-2} \end{cases} \tag{A.8}$$

we obtain that $G_U(\psi_x)-J\psi^2\geqslant 0$ for all $|\psi_x|\geqslant C_U^1(J)$, or in other words

$$q_x(\psi_x)-J\psi_x^2\geqslant\tfrac{1}{4}\psi_x^{2k}\geqslant\tfrac{1}{4}\zeta_x^{2k-2}\psi_x^2 \quad \text{whenever } |\psi_x|>|\zeta_x| \text{ and } |\zeta_x|\geqslant C_U^2(J) \tag{A.9}$$

Collecting together (A.6) and (A.9) and defining

$$C_U=\max\{C_U^1,C_U^2,1\}$$

$$C_U(J)=\begin{cases} C_U & \text{if} \quad J\leqslant\frac{1}{5}C_U^{2k-2} \\ \sqrt[2k-2]{5J} & \text{if} \quad J>\frac{1}{5}C_U^{2k-2} \end{cases} \tag{A.10}$$

we get

$$q_x(\psi_x)-J\psi_x^2 \geqslant \tfrac{1}{4}\zeta_x^{2k-2}\psi_x^2, \quad \text{whenever } |\zeta_x|\geqslant C_U(J) \quad \text{and} \quad \forall\psi_x\in\mathbf{R}$$

and (3.5), which is the first part of the lemma, is proved.

Inequality (3.6) follows trivially from (A.2). For $|\psi_x|\leqslant|\zeta_x|$ we can use (A.2) to obtain

$$q_x(\psi_x)=P_k(\psi_x)+\sum_{i=1}^{k-1} u_{2i}P_i(\psi_x)\leqslant 4^k\zeta_x^{2k-2}\psi_x^2+\sum_{i=1}^{k-1}|u_{2i}|\,4^i\zeta_x^{2i-2}\psi_x^2$$

and for $|\zeta_x|\geqslant C_U(J)\geqslant 1$

$$q_x(\psi_x)\leqslant \zeta_x^{2k-2}\psi_x^2\left[4^k+\sum_{i=1}^{k-1}|u_{2i}|\,4^i\right]\leqslant B_U\zeta_x^{2k-2}\psi_x^2$$

where

$$B_U=4^k+\sum_{i=1}^{k-1}|u_{2i}|\,4^i \tag{A.11}$$

Proof of part ii). We now consider the case in which $|\zeta_x|\leqslant C_U(J)$. Then, using again (A.1)

$$q_x(\psi_x)-J\psi_x^2\geqslant P_k(\psi_x)-\{[C_U(J)]^{2k-2}\,B_U+J\}\,\psi_x^2\geqslant P_k(\psi_x)-D_U(J)\,\psi_x^2$$

where

$$D_U(J)=[C_U(J)]^{2k-2}\,B_U+J=\begin{cases}(5B_U+1)\,J & \text{if } J>\frac{1}{5}C_U^{2k-2}\\ C_U^{2k-2}B_U & \text{if } J\leqslant\frac{1}{5}C_U^{2k-2}\end{cases} \tag{A.12}$$

Hence, using definition (3.3)

$$\begin{aligned}q_x(\psi_x)-J\psi_x^2&\geqslant\tfrac{1}{2}\psi_x^{2k}+\left[\frac{1}{2}\psi_x^{2k}-[C_U(J)]^{2k-2}\,4^k\sum_{j=2}^{2k-1}|\psi_x|^j-D_U(J)\,\psi_x^2\right]\\&\geqslant\tfrac{1}{2}\psi_x^{2k}+g(\psi_x)\end{aligned}$$

where $g(t)=\frac{1}{2}t^{2k}-[C_U(J)]^{2k-2}\,4^k\sum_{j=2}^{2k-1}|t|^j-D_U(J)\,t^2$. Clearly $g(t)$ is bounded below, i.e., we have that

$$g(t)\geqslant[\tfrac{1}{2}t^{2k}-E_U(J)\,t^{2k-1}]\geqslant -4^k[E_U(J)]^{2k}$$

where

$$E_U(J) = \begin{cases} (5B_U + 1 + 2k4^k)\, J & \text{if} \quad J > \frac{1}{5} C_U^{2k-2} \\ (2k4^k + B_U)\, C_U^{2k-2} + J & \text{if} \quad J \leqslant \frac{1}{5} C_U^{2k-2} \end{cases} \tag{A.13}$$

Hence we get

$$q_x(\psi_x) - J\psi_x^2 \geqslant \tfrac{1}{2}\psi_x^{2k} - F_U(J)$$

where

$$F_U(J) = 4^k[E_U(J)]^{2k} = \begin{cases} 4^k[(5B_U + 1 + 2k4^k)]^{2k}\, J^{2k} & \text{if} \quad J > \frac{1}{5} C_U^{2k-2} \\ 4^k[(2k4^k + B_U)\, C_U^{2k-2} + J]^{2k} & \text{if} \quad J \leqslant \frac{1}{5} C_U^{2k-2} \end{cases} \tag{A.14}$$

note that $F_U(J)$ is increasing proportionally to J^{2k} for J sufficiently large and is bounded below by a constant greater than 1.

On the other hand, in a completely analogous way, it is simple to show that $q_x(\psi_x) \leqslant 2\psi_x^{2k} + F_U(J)$.

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